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NUMERICAL ADVECTION EXPERIMENTS [1]

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ABSTRACT

First, second, and fourth order finite difference approximations to the color equation in both advection and conservation form are considered in one and two space dimensions. All schemes considered are based on forward time differences and most involve centered space differences. All are shown to be numerically stable for $|u\Delta t/\Delta x| \leq 1$. Test calculations indicate that for the same order of accuracy, the conservation form produces more accurate solutions than the advection form. For either conservation or advection form, fourth order schemes are shown to be more accurate than second or first order schemes in terms of both amplitude and phase errors.

1. INTRODUCTION

Both analytic and numerical solutions of the Eulerian equations of hydrodynamics are limited in extent by the nonlinear advection, or transport terms. Analytic solutions are difficult to obtain because the advection terms render the equations nonlinear. Numerical solutions are readily secured in principle but are inaccurate because finite difference approximations of the advection terms can introduce errors in both phase and amplitude. This paper deals with several different methods of numerically modeling the advective process.

The continuity equation and the Euler equations, for flows with neither sources, sinks, nor body forces, are

$$\frac{\partial \rho}{\partial t}+(\rho u^i)_{,i}=0 \tag{1}$$

$$\frac{\partial u_k}{\partial t}+u^i u_{k,i}=-\frac{1}{\rho}p_{,k}. \tag{2}$$

Equation (1) may be combined with (2) to give

$$\frac{\partial \rho u_k}{\partial t}+(\rho u^i u_k)_{,i}=-p_{,k}. \tag{3}$$

Equations (1) and (3) are in the so-called conservation form; mass and momentum transports appear as divergences of mass and momentum fluxes. Green's theorem may be used to transform these terms to surface integrals, and this leads to particularly satisfying finite difference approximations in which mass and momentum are identically conserved [8].

Equation (2) is in advection form. Although based on reasonable physical arguments, finite difference approximations to the advective vector ($u^i u_{k,i}$) do not necessarily contain momentum conservation.

Equations (1) and (2) describe the same physical situation as equations (1) and (3); for identical boundary and initial conditions, the two sets must give the same exact solution. Since there is no unique way of writing finite difference approximations to partial derivatives, the numerical solutions from the two sets of difference equations are expected to be different. They must, however, approach the exact solution as Δt and Δx approach zero.

Roberts and Weiss [11] have examined second and fourth order numerical schemes for equations in conservation form. Bryan [1] has pointed out that a numerical nonlinear instability can be eliminated by properly differencing equations in conservation form. The well known Lax-Wendroff scheme [4] is based on the equations being in conservation form, and has been used with remarkable success by Burstein [2] in two dimensional compressible hydrodynamic calculations.

In order to obtain the advantages offered by conservation form, the finite difference equations must be properly composed with respect to accuracy, numerical stability, and conservation. In certain coordinate systems, however, proper evaluation of the transport terms may lead to quite complicated algebraic forms that can be relatively time consuming to compute. On the other hand, the advection

[1] This work was performed under the auspices of the U.S. Atomic Energy Commission.

form may give simpler and thus cheaper, difference equations. Even if one is then forced to use the advection form rather than the conservation form by practical matters, such as limited available computer time, it is possible that the trade-off of accuracy for efficiency will not cause the results to be utterly unreliable. Although the advection form of the equations does not guarantee conservation, it does not follow that the proper quantities will not be approximately conserved.

The dependent variables in the partial differential equations of interest are generally all functions of space and time, so that both time derivatives and space derivatives must be numerically approximated. Since the evolution of these quantities in time is desired, the usual approach is to use space derivatives to evaluate time derivatives. This may be accomplished by implicit methods [6], centered methods [10], or so-called one-sided methods. In the (forward) one-sided method, values of the space derivative at time t are used to estimate the time derivative at $t+\Delta t/2$ and to thus advance the state of the system to time $t+\Delta t$. Space derivatives may be either one-sided, or centered, and examples of both will be given, but centered space derivatives and forward time derivatives will be of primary interest.

The intent of this paper is thus to examine several different finite difference approximations to the advective process. First, second, and fourth order schemes for equations in both advection and conservation form are analyzed. It is shown that all schemes are stable for $|u\Delta t/\Delta x|\leq 1$, but that each scheme introduces both phase and amplitude errors. These errors, for the linear case, can be calculated and are displayed as contour plots. Results of test calculations in one and two space dimensions are also given.

2. THE COLOR EQUATION

In order to examine the problems of numerical advection, finite difference approximations to the color equation [2]

$$\frac{\partial\psi}{\partial t}+u\frac{\partial\psi}{\partial x}+v\frac{\partial\psi}{\partial y}=0 \tag{4}$$

in Cartesian coordinates will be analyzed, where $\psi=\psi(x, y, t)$. An equivalent form of (4) is

$$\frac{\partial\psi}{\partial t}+\frac{\partial\psi u}{\partial x}+\frac{\partial\psi v}{\partial y}-\psi\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0. \tag{5}$$

The dependent variable ψ is some property (nondiffusive color, say) of the fluid that is transported along by the flow field so that its total derivative along an instantaneous streamline is zero. That is, equations (4) and (5) when written with respect to an observer who moves with the fluid simplify to

$$\frac{d\psi}{dt}=0, \qquad \psi=\psi(x_0, y_0, t)$$

[2] Equation (4) was named the "color" equation in the early 1950's by R. Lelevier who noted that ψ has the properties of a nondiffusive color.

so that the observer will measure no change in ψ as time passes. This notion will be used in forming the difference equations for the advection form of the color equation.

Equation (4) is the advection form of the color equation and equation (5) is the conservation form of that equation. It is emphasized that this equation is written in two different forms because each lends itself to a finite difference approximation based on a particular physical argument, but that the same solution is expected from both cases.

For simplicity u and v, the components of velocity in the x and y directions, respectively, will be assumed to be specified functions of space but independent of time although in general they may be time dependent also.

3. ONE DIMENSIONAL ADVECTION FORMULATION

The color equation in advection form in one space dimension is

$$\frac{\partial\psi}{\partial t}+u\frac{\partial\psi}{\partial x}=0$$

where now $\psi=\psi(x, t)$.

If u is a constant, and $\psi(x, 0)=f(x)$, a solution is

$$\psi(x, t)=f(x-ut) \tag{6}$$

and this furnishes the basis for the following finite difference approximation.

Let $\psi_j^N=\psi(j\Delta x, N\Delta t)$ where j and N are integers. ψ_j^N is then known at the intersection of a space-time lattice. Given ψ_j^N, for all j, in order to compute ψ_j^{N+1}, one constructs a characteristic (of slope $\delta x/\delta t=u$) through the points $(j\Delta x, (N+1)\Delta t)$ and $((j+r)\Delta x, N\Delta t)$ where r is not necessarily an integer (fig. 1). Equation (6) then states

$$\psi(j\Delta x, (N+1)\Delta t)=\psi^*=\psi((j+r)\Delta x, N\Delta t)$$

where $\delta x=-r\Delta x$. Since r is not in general an integer it is necessary to interpolate on the ψ_j^N field to determine ψ^*. Interpolation on the three points $j-1$, j, $j+1$ results in

$$\psi^*=\psi_j^N-\frac{\alpha}{2}(\psi_{j+1}^N-\psi_{j-1}^N)+\frac{\alpha^2}{2}(\psi_{j+1}^N-2\psi_j^N+\psi_{j-1}^N) \tag{7}$$

where $\alpha=u\Delta t/\Delta x$, and ψ is thus advanced in time by setting

$$\psi_j^{N+1}=\psi^*.$$

Limiting this to an interpolation procedure requires that $|\alpha|<1$.

The above result may also be obtained by a Taylor series expansion in time, which will display the order of the truncation errors

$$\psi_j^{N+1}=\psi_j^N+\left.\frac{\partial\psi}{\partial t}\right|_j^N\Delta t+\left.\frac{\partial^2\psi}{\partial t^2}\right|_j^N\frac{\Delta t^2}{2}+O(\Delta t^3).$$

Using the differential equation to convert time derivatives to space derivatives yields

$$\psi_j^{N+1}=\psi_j^N-u\Delta t\left.\frac{\partial\psi}{\partial x}\right|_j^N+\frac{u^2\Delta t^2}{2}\left.\frac{\partial^2\psi}{\partial x^2}\right|_j^N-\frac{\Delta t^2}{2}\left.\frac{\partial\psi}{\partial x}\right|_j^N\left.\left(\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial x}\right)\right|_j^N$$

and evaluation of $\partial\psi/\partial x$ and $\partial^2\psi/\partial x^2$ to second order gives

$$\psi_j^{N+1}=\psi_j^N-\frac{\alpha}{2}(\psi_{j+1}^N-\psi_{j-1}^N)+\frac{\alpha^2}{2}(\psi_{j+1}^N-2\psi_j^N+\psi_{j-1}^N)+O(\alpha\Delta x^3)+O(\Delta t^2) \quad (8)$$

where the term $O(\Delta t^2)$ occurs in the general case where $u=u(x, t)$. Thus the evaluation of ψ^{N+1} by (8) has errors of order Δt^2 and Δx^3 and this is usually referred to as a second order scheme, although it is only first order accurate in time.

In order to make the notation less cumbersome, spatial differences will from now on be written in terms of linear difference operators which will be represented by capital roman letters. Schemes based on forward time differences are thus in general written

$$\psi_j^{N+1}=(\mathrm{I}\psi^N)_j-(\mathrm{A}\psi^N)_j=[(\mathrm{I}-\mathrm{A})\psi^N]_j \quad (9)$$

where ψ is a vector with components $\psi_1, \psi_2, \ldots, \psi_J$; A is the spatial difference operator, and $(\mathrm{I}\psi)_j=\psi_j$. The particular scheme just discussed (the quadratic advection Scheme) has the operator

$$(\mathrm{A}_2\psi)_j=\frac{\alpha}{2}(\psi_{j+1}-\psi_{j-1})-\frac{\alpha^2}{2}(\psi_{j+1}-2\psi_j+\psi_{j-1}). \quad (10)$$

If the interpolation for ψ^* is carried out on a curve obtained by fitting the five points surrounding j with a polynomial, a "fourth order" scheme is produced with the operator.

$$(\mathrm{A}_4\psi)_j=\frac{\alpha}{12}[8(\psi_{j+1}-\psi_{j-1})-(\psi_{j+2}-\psi_{j-2})]$$
$$+\frac{\alpha^2}{24}[30\psi_j-16(\psi_{j+1}+\psi_{j-1})+(\psi_{j+2}+\psi_{j-2})]$$
$$+\frac{\alpha^3}{12}[-2(\psi_{j+1}-\psi_{j-1})+(\psi_{j+2}-\psi_{j-2})]$$
$$-\frac{\alpha^4}{24}[6\psi_j-4(\psi_{j+1}+\psi_{j-1})+(\psi_{j+2}+\psi_{j-2})]. \quad (11)$$

Thus

$$(\mathrm{A}_4\psi)_j=\alpha\Delta x\frac{\partial\psi}{\partial x}-\frac{(\alpha\Delta x)^2}{2}\frac{\partial^2\psi}{\partial x^2}+\frac{(\alpha\Delta x)^3}{3!}\frac{\partial^3\psi}{\partial x^3}-\frac{(\alpha\Delta x)^4}{4!}\frac{\partial^4\psi}{\partial x^4}$$
$$-O(\alpha\Delta x^5)+O(\alpha^2\Delta x^6)-O(\alpha^3\Delta x^5)+O(\alpha^4\Delta x^6).$$

Since $\alpha\sim1$, this operator is then fourth order accurate in space but since the time and spatial variation in u are still not accounted for, it also has errors $O(\Delta t^2)$ as does the previous scheme..

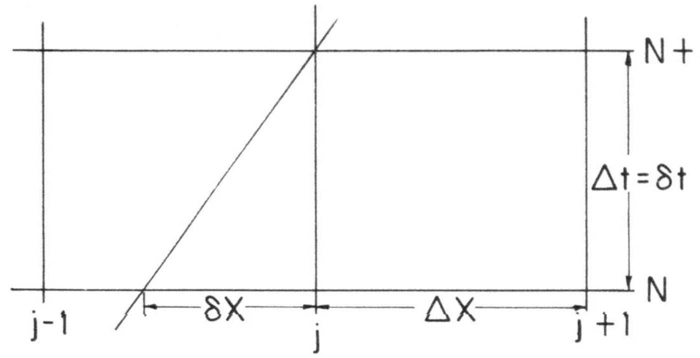

FIGURE 1.—Advection diagram.

For completeness, two more types of schemes are given below. The first originated with Lelevier [9] and is an example of a first order scheme. It contains an appreciable amount of numerical damping, which makes it unsuitable for problems involving relatively long integration times.

$$(\mathrm{A}\psi)_j=(u_j-|u_j|)(\psi_{j+1}-\psi_j)\frac{\Delta t}{2\Delta x}+(u_j+|u_j|)(\psi_j-\psi_{j-1})\frac{\Delta t}{2\Delta x}. \quad (12)$$

Thus the scheme involves choosing a one-sided space derivative of ψ from the "upstream" direction. It was used in some early (1958) two dimensional compressible hydrodynamic calculations, but was discarded soon thereafter for a scheme in conservation form [8] in which, however, the one-sided space differences were retained.

The second type involves third order errors in time [10, 3]. These schemes may for example involve two step processes in which lower order intermediate results are first calculated at $t+\Delta t/2$. These and the function at time t are then combined to give the final, higher order result at $t+\Delta t$.

The process is represented symbolically by,

$$\psi_j^{N+1}=[(\mathrm{I}-\mathrm{C})\psi^N]_j$$

where

$$\mathrm{C}=\mathrm{B}(\mathrm{I}-\mathrm{A}),\ (\mathrm{B}\psi)_j=\frac{\alpha}{2}(\psi_{j+1}-\psi_{j-1})$$

and

$$(\mathrm{A}\psi)_j=\frac{1}{2}(\mathrm{B}\psi)_j-\frac{\alpha^2}{8}(\psi_{j+1}-2\psi_j+\psi_{j-1}) \quad (13)$$

which results in a scheme second order accurate in both space and time.

4. ONE DIMENSIONAL CONSERVATION FORMULATION

The conservation form of the color equation in one space dimension is

$$\frac{\partial\psi}{\partial t}+\frac{\partial\psi u}{\partial x}-\psi\frac{\partial u}{\partial x}=0.$$

If the term accounting for the transport of ψ by the fluid, $\partial\psi u/\partial x$, is thought of as the divergence of a flux, application of Green's theorem [7] then predicts that in any zone the decrease of ψ with time is proportional to the net flux out of the zone (modified in this case by the compression term $\psi\partial u/\partial x$).

The flux across a zone boundary at $j+½$ is

$$(\psi u)_{j+1/2}=F_{j+1/2}=\frac{1}{\Delta t}\int_{x_{j+1/2}-u\Delta t}^{x_{j+1/2}}\psi(x')dx'. \tag{14}$$

Assuming ψ to vary linearly between ψ_j and ψ_{j+1} and integrating equation (14) results in

$$F_{j+1/2}\frac{\Delta t}{\Delta x}=\frac{\alpha_{j+1/2}}{2}(\psi_{j+1}+\psi_j)-\frac{\alpha^2_{j+1/2}}{2}(\psi_{j+1}-\psi_j). \tag{15}$$

If ψ is assumed to fit a cubic through $j-1$, j, $j+1$, $j+2$, evaluation of (14) gives

$$\begin{aligned}F_{j+1/2}\frac{\Delta t}{\Delta x}=\frac{\alpha}{16}&[9(\psi_{j+1}+\psi_j)-(\psi_{j+2}+\psi_{j-1})]\\ -\frac{\alpha^2}{48}&[27(\psi_{j+1}-\psi_j)-(\psi_{j+2}-\psi_{j-1})]\\ -\frac{\alpha^3}{12}&[(\psi_{j+1}+\psi_j)-(\psi_{j+2}+\psi_{j-1})]\\ +\frac{\alpha^4}{24}&[3(\psi_{j+1}-\psi_j)-(\psi_{j+2}-\psi_{j-1})]\end{aligned} \tag{16}$$

where $\alpha=\alpha_{j+1/2}$.

A second order flux divergence scheme is then given by

$$\psi_j^{N+1}=\psi_j^N\left[1+\frac{\Delta t}{\Delta x}(u_{j+1/2}-u_{j-1/2})\right]-\frac{\Delta t}{\Delta x}(F^N_{j+1/2}-F^N_{j-1/2}) \tag{17}$$

where $F_{j+1/2}$ is given by (15), and a fourth order flux divergence scheme is

$$\begin{aligned}\psi_j^{N+1}=\psi_j^N\Big[1+\frac{\Delta t}{24\Delta x}\{27(u_{j+1/2}-u_{j-1/2})\\ -(u_{j+3/2}-u_{j-3/2})\}\Big]-\frac{\Delta t}{\Delta x}(F^N_{j+1/2}-F^N_{j-1/2})\end{aligned} \tag{18}$$

where $F_{j+½}$ is given by (16) and $\partial u/\partial x|_j$ has been evaluated to fourth order.

It is clear that the transport term leads to identical conservation of ψ since the only net flux into the total system occurs at the boundaries, all interior fluxes balancing out.

It is seen by comparing equations (15) and (16) with (10) and (11) that in a Cartesian coordinate system at least, the advection and conservation forms require approximately the same number of arithmetic operations per point. This is probably true for scalars in most coordinate systems. Unfortunately, correctly approximating the divergence of a tensor on a nonrectilinear coordinate system involves an increase in the number of arithmetic operations necessary per point, and so in some cases it may be necessary for reasons of economy to use the advection form for the momentum equation. An example of this is the momentum equation on a spherical coordinate system.

A count of arithmetic operations shows that the second order advection operator requires 5 multiplies and 6 additions per point while the fourth order advection operator requires 13 multiplies and 18 additions per point. The increased accuracy thus costs a factor of three in computing speed.

5. STABILITY ANALYSIS

A solution of the color equation is

$$\psi(x,t)=e^{ik(x-ut)}=\psi(x,0)e^{-ikut}$$

so that the initial configuration $\psi(x,0)$ is merely translated a distance ut in time t and the solution after a time interval Δt has a phase angle $-ku\Delta$t$=-\theta\alpha$ where $\theta=k\Delta x$. There is no amplitude damping.

For the difference equations, phase and amplitude errors as well as a necessary condition for stability are given by the eigenvalues of (I$-$A) in equation (9).[3] Substitution of an eigenvector $e^{ikj\Delta x}$ for ψ_j in (9) results in the recursion $\psi^{N+1}=\xi(k)\psi^N$ where $\xi(k)$ is a complex eigenvalue of a particular difference operator. For the difference equations considered here, which employ one-sided time derivatives, $|\xi|\neq1$ so that an amplitude modification occurs each cycle. Since stability dictates that $|\xi|\leq1$ and the analytic solution involves no amplitude damping, it is desirable for ξ to be as close to the unit circle (but not outside) as possible. The phase angle is, from the numerical solution in time Δt,

$$\delta=\tan^{-1}[\mathrm{Im}\,(\xi)/\mathrm{Re}\,(\xi)]$$

and in general $\delta\neq-\theta\alpha$. Thus the relative errors in amplitude and phase in a time increment Δt are $1-|\xi|$ and $1-\delta/\theta\alpha$, respectively.

Substituting $e^{ikj\Delta x}$ for ψ_j into (10) and (11) in turn, it is readily seen that the eigenvalues of I$-$A are

$$\xi_2=1-\alpha^2(1-\cos\theta)-i\alpha\sin\theta \tag{19}$$

and

$$\begin{aligned}\xi_4=1-\frac{\alpha^2}{12}(15-16\cos\theta+\cos2\theta)+\frac{\alpha^4}{12}(3-4\cos\theta+\cos2\theta)\\ -i\left\{\frac{\alpha}{6}(8\sin\theta-\sin2\theta)+\frac{\alpha^3}{6}(-2\sin\theta+\sin2\theta)\right\}.\end{aligned} \tag{20}$$

The magnitude of ξ_2 is then

$$|\xi_2|^2=1-\alpha^2(1-\alpha^2)(1-\cos\theta)^2$$

[3] If the matrix of B is normal, then a necessary and sufficient condition for the stability of $\psi^{N+1}=B\psi^N$ is $|r|\leq1$ where r is the maximum eigenvalue of B [9]. If B is not normal, then $|r|\leq1$ is necessary but not sufficient for stability. In practice B is rarely normal but it is found that the restriction $|r|\leq1$ usually prevents the occurrence of instabilities.

so that stability is guaranteed by $\alpha^2 \leq 1$. The magnitude of ξ_4 is

$$|\xi_4|^2 = 1 - \frac{\alpha^2}{36}(1-\alpha^2)(1-\cos\theta)^3 f(\alpha,\theta)$$

where

$$f(\alpha,\theta) = (1-\cos\theta)\alpha^4 - (9-5\cos\theta)\alpha^2 + 4(5-\cos\theta).$$

Let $F(\alpha) = f(\alpha,\omega)$ where ω is any (fixed) value of θ. The roots of $F(\alpha)$ are then ± 2 and $\pm\sqrt{(5-\cos\omega)/(1-\cos\omega)}$. From this it is seen that since $F(\alpha) > 0$ for $\alpha^2 < 3$, the fourth order advection scheme is stable for $\alpha^2 \leq 1$.

Evaluation of the eigenvalues of the operators used in the conservation form is accomplished by assuming α to be a constant which then reduces the conservation equations to (9). The eigenvalues for the second order scheme turn out to be the same for conservation and advection forms, but the fourth order-conservation form has the eigenvalues,

$$\xi_{4c} = 1 - \frac{\alpha^2}{24}(27 - 28\cos\theta + \cos 2\theta) + \frac{\alpha^4}{12}(3 - 4\cos\theta + \cos 2\theta)$$
$$-i\left\{\frac{\alpha}{8}(10\sin\theta - \sin 2\theta) + \frac{\alpha^3}{6}(-2\sin\theta + \sin 2\theta)\right\}. \quad (21)$$

The magnitude of ξ_{4c} is

$$|\xi_{4c}|^2 = 1 - \frac{\alpha^2}{144}(1-\cos\theta)^2 g(\alpha,\theta)$$

where

$$g(\alpha,\theta) = 87 - 72\cos\theta + 9\cos^2\theta - \alpha^2(1-\cos\theta)(97-25\cos\theta)$$
$$+4\alpha^4(1-\cos\theta)(9-5\cos\theta) - 4\alpha^6(1-\cos\theta)^2.$$

It can be shown that $|\xi_{4c}|^2$ has extrema at $\theta = 0$ and π, but since $|\xi_{4c}|^2 \equiv 1$ for $\theta = 0$, the extremum at $\theta = \pi$ is the one of concern since it can result in numerical instabilities. (It is usually true that the higher wave numbers are the least stable.)

Evaluation of $g(\alpha,\theta)$ at $\theta = \pi$ results in

$$g(\alpha,\pi) = 16(1.5-\alpha^2)(2-\alpha^2)(3.5-\alpha^2)$$

so that the fourth order conservation scheme is stable for $\alpha^2 \leq 1.5$.

In order to further examine the relationship of the solution obtained by difference approximations to the analytic solution, contour plots of $|\xi|$ and of relative phase as functions of θ/π and α have been constructed (figs. 2–11). In these figures, the contour interval is 0.1; functional values less than one are plotted as broken lines while solid lines represent values greater than or equal to one.

Figures 2 through 6 present level lines of $|\xi|$ for the several difference schemes discussed here. Since dotted lines cover numerically stable regions it is seen that all schemes are stable for all wave numbers if $|\alpha| \leq 1$. Further, the fourth order conservation scheme is stable for $\alpha^2 < 1.5$ while the two-step second order scheme is stable if $|\alpha| \leq 2$, and has very little damping for $|\alpha| < 1$.

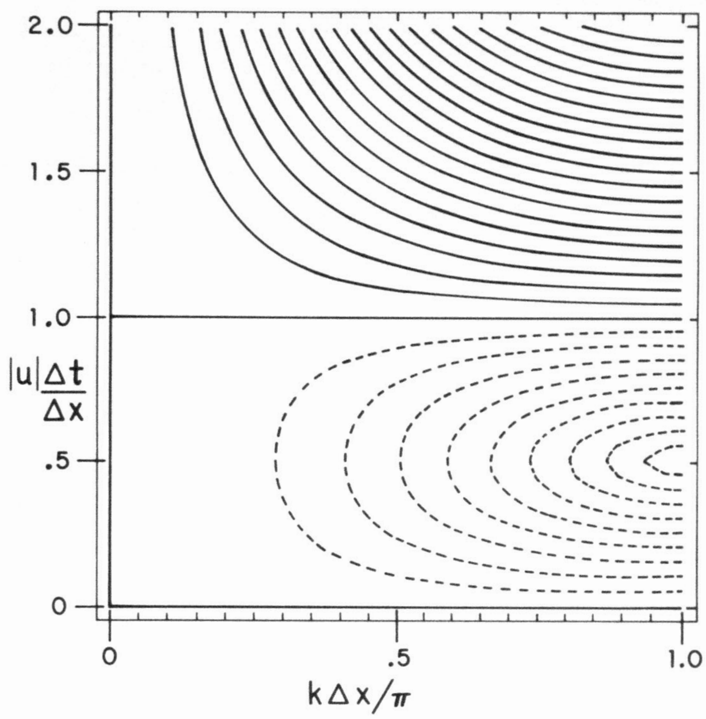

FIGURE 2.—Level lines of $|\xi|$ for equation (12) (first order) as a function of θ/π and α. The contour interval is 0.1; broken lines represent constant values of $|\xi| < 1$ and solid lines represent constant values of $|\xi| \geq 1$. Figures 2 through 11 all use the same plotting convention.

Figures 7 through 11 are level lines of the relative phase, $-\delta/\theta\alpha$ as a function of wave number and α as computed for the difference schemes discussed here. It is seen that for most schemes, in the stable region, the contours are broken lines indicating that waves in the numerical solution move more slowly than they analytically should.

The figures for one step forward time differences all have the same characteristics. For a fixed value of α, both amplitude and phase errors increase as θ increases, becoming maximum for $\theta = \pi$. This is the highest wave number a given mesh can support and this wave has a zero relative phase velocity. Fortunately, maximum damping occurs at this same wave number so that the numerical amplitude error compensates for the relative phase error in this case. For a given wave number the amplitude damping goes through a maximum as α runs from zero to one. In a perverse manner, the phase error goes through a minimum for the same range of α, so that once again the damping error compensates for the phase error, but in an undesirable way this time. For low wave numbers, the phase error is less for $\alpha = 1$ than for $\alpha = 0$. Thus calculations for which α is small will suffer large phase errors but less than maximum damping, and calculations for which α is near unity will also have less than maximum damping but will have smaller phase errors. Intermediate values will result in small phase errors, but large amplitude errors.

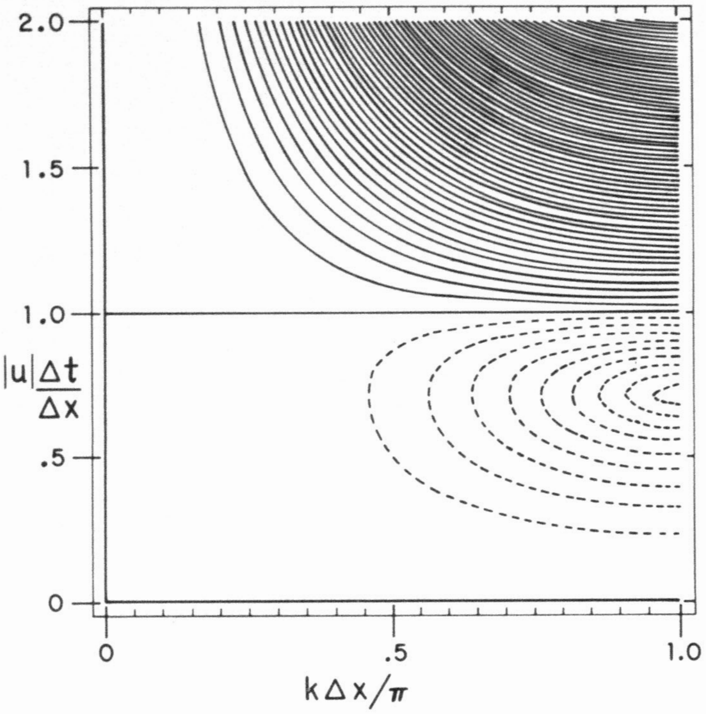

FIGURE 3.—Level lines of $|\xi|$ for equation (17) (linearized) or (10) and (9) (second order).

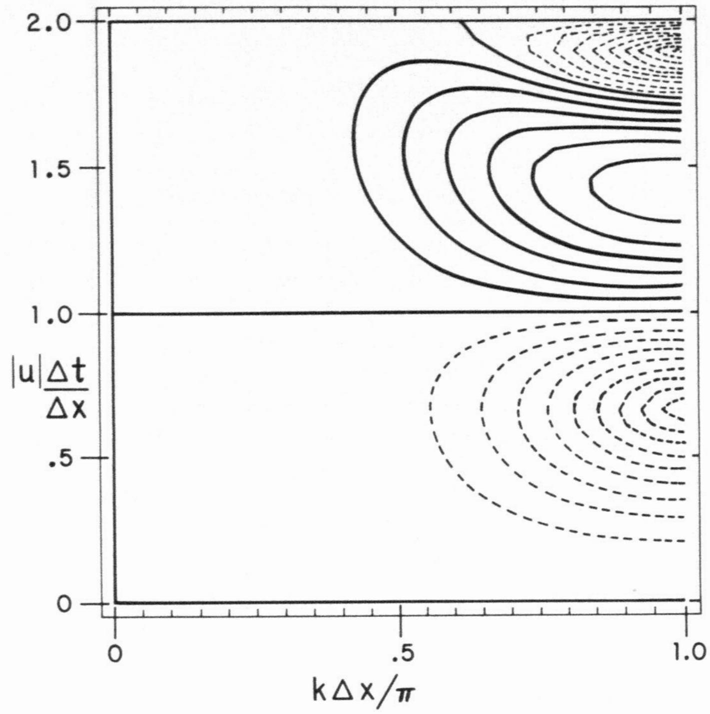

FIGURE 5.—Level lines of $|\xi|$ for equations (11) and (9) (fourth order).

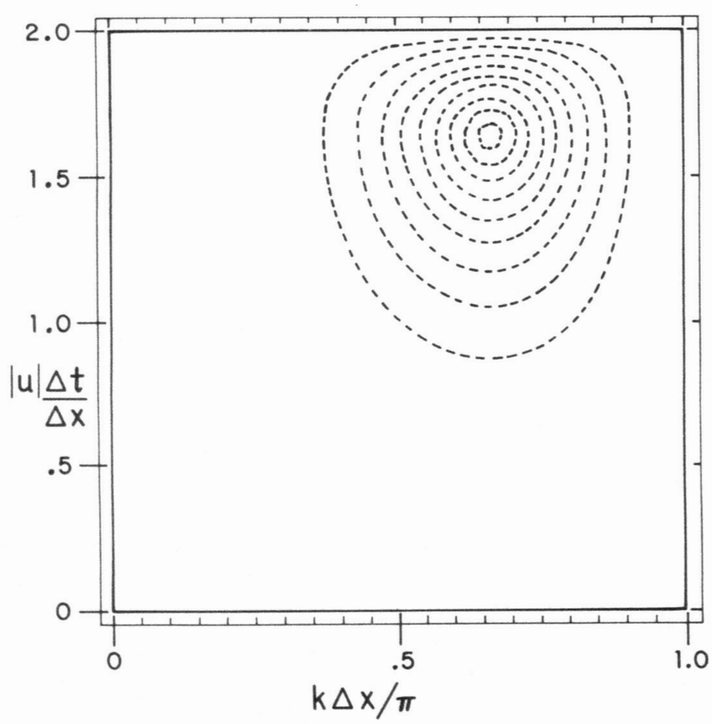

FIGURE 4.—Level lines of $|\xi|$ for equation (13) (second order, two-step).

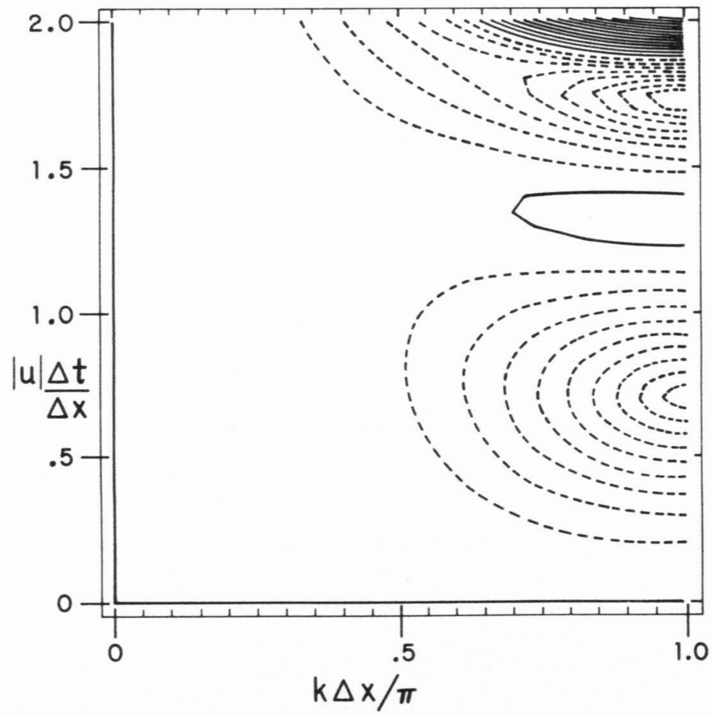

FIGURE 6.—Level lines of $|\xi|$ for equation (18) (fourth order, conservation).

It is seen that the fourth order conservation form (fig. 11) has a greater relative phase error than the fourth order advection form (fig. 10), but that it still is an improvement over lower order schemes.

Comparison of figures 2 and 3 shows that the second order scheme has appreciably less damping than the first order scheme, but figures 7 and 8 indicate that the second order scheme has a slightly greater phase error

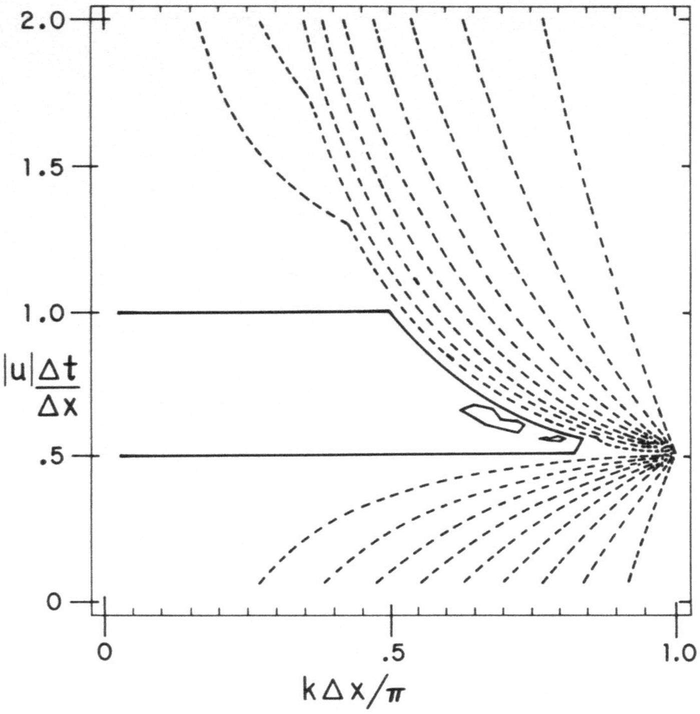

FIGURE 7.—Level lines of $-\delta/\theta\alpha$, the relative phase for equation (12) (first order). (For constant u, the correct phase is $-\theta\alpha$. The phase for solutions generated by the difference scheme is δ.)

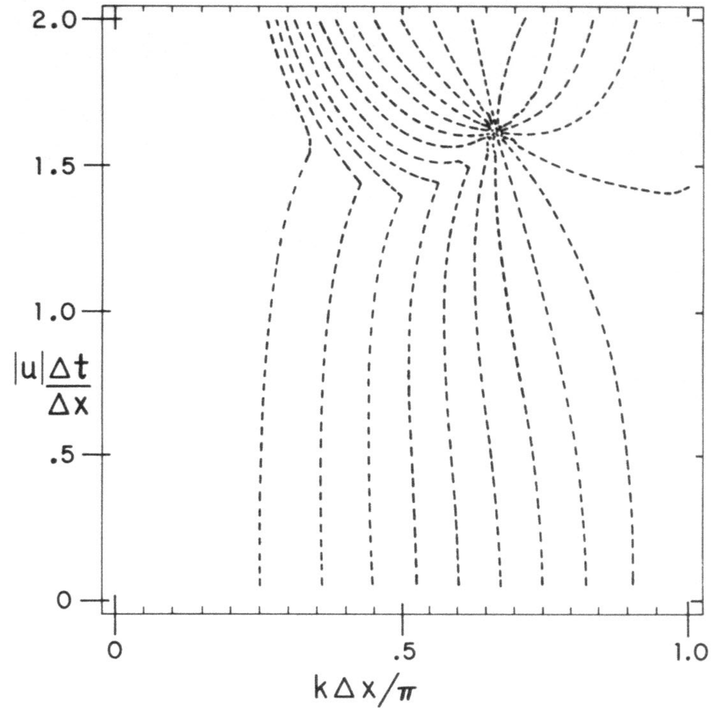

FIGURE 9.—Level lines of $-\delta/\theta\alpha$ for equation (13) (second order, two-step).

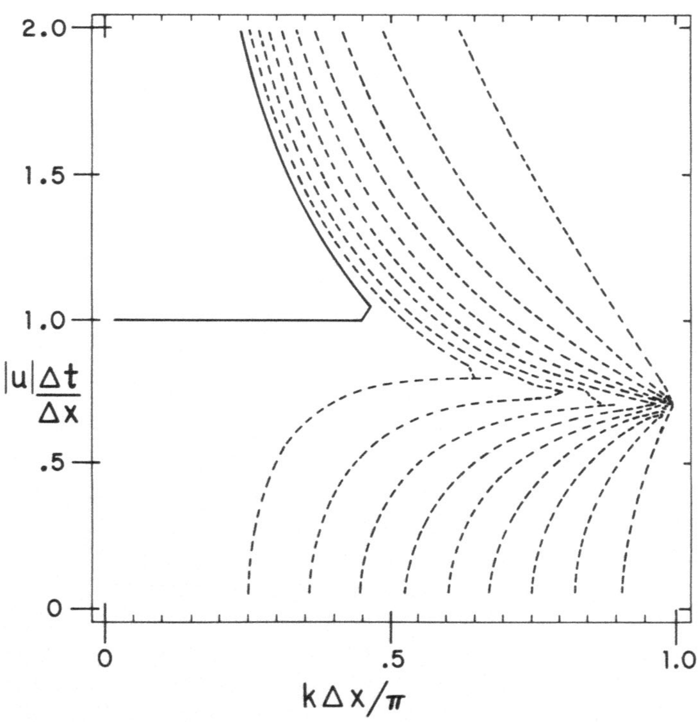

FIGURE 8.—Level lines of $-\delta/\theta\alpha$ for equation (17) (linearized) or (10) and (9) (second order).

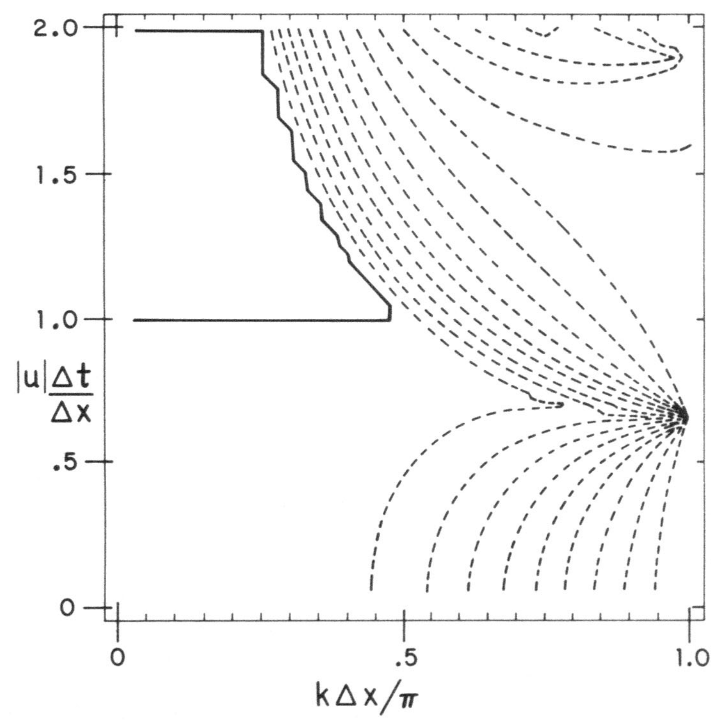

FIGURE 10.—Level lines of $-\delta/\theta\alpha$ for equations (11) and (9) (fourth order).

at each wave number (for low α) which becomes larger as α increases.

The two-step procedure (compare fig. 9 with fig. 8) tends to linearize the phase error in that it is almost independent of α, at least for $\alpha \leq 1$. This then does not improve the accuracy, but does tend to make the phase error more predictable.

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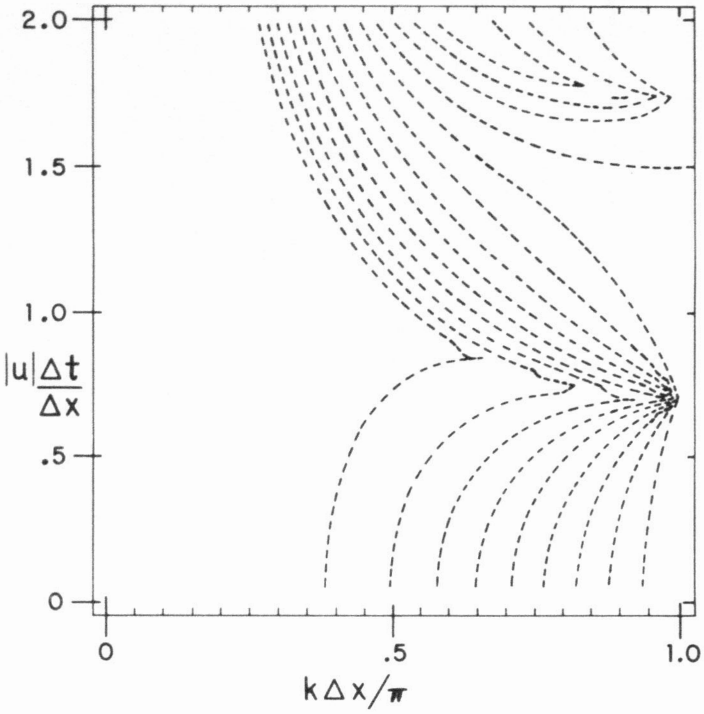

FIGURE 11.—Level lines of $-\delta/\theta\alpha$ for equation (18) (fourth order, conservation).

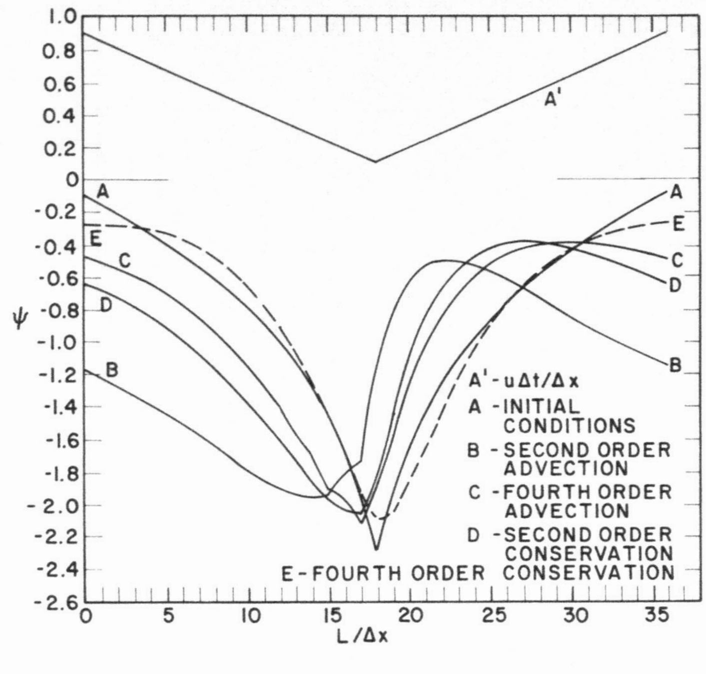

FIGURE 12.—Results of one dimensional test calculations with second and fourth order schemes.

6. ONE DIMENSIONAL CALCULATIONS

It can be shown that given $\psi(x,0)=f(\log u)$, $u=ax+b$ (a and b constants), a solution of

$$\frac{\partial \psi}{\partial t}+u\frac{\partial \psi}{\partial x}=0$$

is

$$\psi(x,t)=f(\log u-at).$$

Thus given a flow in which the divergence of the velocity field is a constant in space and time and the initial ψ configuration is any function of the logarithm of that velocity field, the value of $\psi(x,t)$ is determined for $t\geq 0$.

This solution differs from the usual wave equation type of solution (equation (6)) in that the velocity need not be a constant. Since in the test problems the velocity field is chosen not to be a constant in space, the term $\psi\, \partial u/\partial x$ does not drop out of the conservation form and a realistic test of advection versus conservation is achieved.

The test calculation used to obtain solutions from the various difference schemes is the following. In the interval $0\leq x\leq L/2$, $u=0.9-1.6x/L$ and for $L/2\leq x\leq L$, $u=-0.7+1.6x/L$ where L is some length. Initial conditions are generated from the u field to be $\psi(x,0)=\log u$. Periodic spatial boundary conditions, $\psi(0,t)=\psi(L,t)$, are imposed.

Figures 12 and 13 give results of one dimensional test calculations from various difference schemes. In both figures, curve A′ is a plot of $u\Delta t/\Delta x$, and curve A is $\psi(x,0)$, the initial ψ configuration for all tests. In all calculations, $\Delta t/\Delta x=1$ so that α varies between 0.1 and 0.9.

Figure 12 gives the results of one dimensional calculations with the second and fourth order schemes for both the advection and conservation forms. Each solution is the result of 989 calculation cycles which corresponds to the curve being translated through a distance $10L$ or 360 zones. Curve B is the solution generated by a combination of equations (10) and (9), the quadratic advection scheme. Curve C is also a result of the advection prescription, being the fourth order solution, a combination of equations (11) and (9). Curve D is the solution of the second order conservation form (equation (17)) and curve E is the fourth order conservation form solution (equation (18)).

Figure 13 gives results of test calculations with some odd schemes. Curve B is the result of a two step, second order calculation (equation (13)) and curve C is the result of a fourth order two-step calculation similar to equation (13) but with the second order spatial operators replaced by fourth order operators. Both of these curves are plotted after 989 cycles so that the curve has moved a distance $10L$. Curve D is the result of a first order calculation in conservation form using the ideas contained in equation (12). It is the result of only 99 cycles so that this curve has only been transported a distance L. After 989 cycles, the strong damping in this scheme has reduced the result to $\psi=-1.2$ for all x.

Comparing curves B and C in figures 12 and 13 it is seen that the two-step calculations do result in slightly more accurate solutions, but that the accuracy is improved even more by switching from advection to conservation form, the most accurate result coming from the fourth order scheme in conservation form.

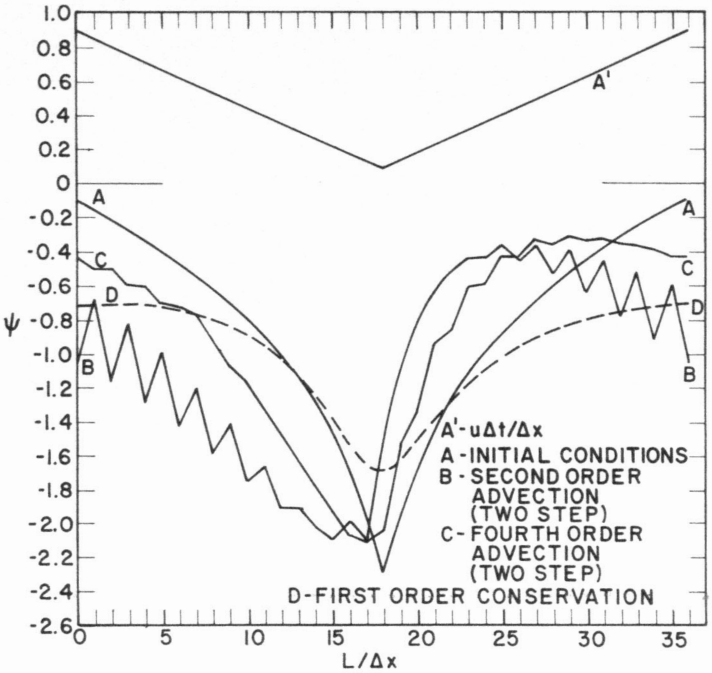

FIGURE 13.—Result of one dimensional test calculations with second and fourth order two-step schemes and a first order conservation scheme.

In experiments, it has been found in all the test cases considered in this report, that an abbreviated fourth order scheme (advection form) gives results which differ very little from those obtained with (11). The abbreviated form is produced by dropping the last two terms in (11), the α^3 and α^4 terms.

7. TWO DIMENSIONAL CALCULATIONS

In two dimensions, the color equation is

$$\frac{\partial\psi}{\partial t}+u\frac{\partial\psi}{\partial x}+v\frac{\partial\psi}{\partial y}=0$$

and $\psi(x, y, t)$ is assumed to be known on a three dimensional space-time mesh where $\psi^N_{k,l}=\psi(k\Delta x,\ l\Delta y,\ N\Delta t)$ for integral values of k, l, N.

Difference equations in two dimensions are constructed following Marchuk's method of fractional time steps [5, 6], so that in operator notation, the advection form becomes,

$$\psi^{N+1}_{k,l}=[(\mathrm{I}-\mathrm{A})(\mathrm{I}-\mathrm{B})\psi^N]_{k,l}$$

where $\mathrm{A}\psi$ and $\mathrm{B}\psi$ represent advection in the x and y directions. In practice, ψ^{N+1} is the result of two distinct computations; first $\psi^+=(\mathrm{I}-\mathrm{B})\psi^N$, followed by $\psi^{N+1}=(\mathrm{I}-\mathrm{A})\psi^+$. A and B are thus one dimensional operators and the two dimensional schemes (both advection and conservation forms) considered here are in fact composed of the one dimensional operators discussed in earlier sections of this report.

The approximation to the conservation form is written

$$\psi^{N+1}_{k,l}=[(1+r)(\mathrm{I}-\mathrm{A})(\mathrm{I}-\mathrm{B})\psi^N]_{k,l}$$

where $\mathrm{A}\psi$ and $\mathrm{B}\psi$ are proportional to fluxes in the x and y directions and r is the appropriate finite difference estimate of $\partial u/\partial x+\partial v/\partial y$, the divergence of the velocity field ($1+r$ is a scalar).

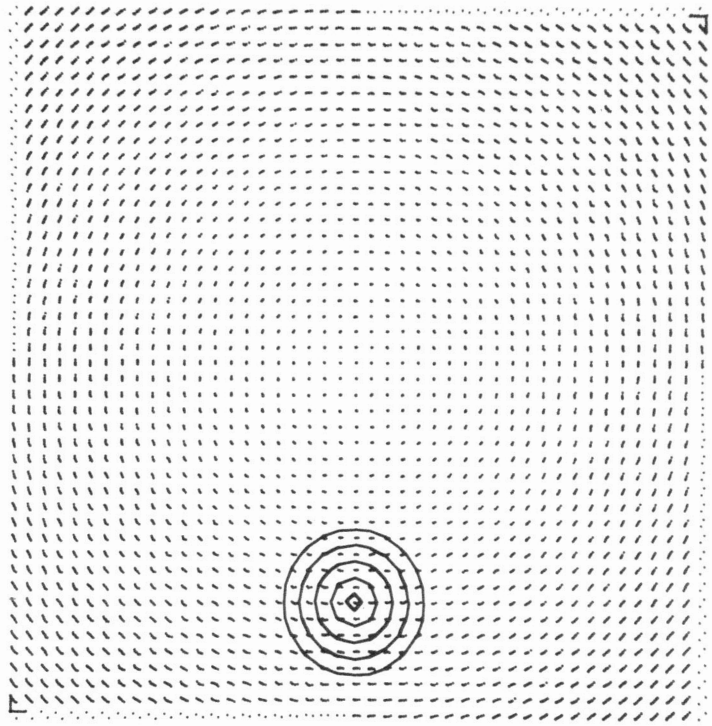

FIGURE 14.—Initial conditions for two dimensional calculations.

The stability analysis of two dimensional schemes is made quite simple if the method of fractional time steps is adopted, for then the stability of the total step is guaranteed if each separate step is itself stable. Thus only the component one dimensional operators have to be examined for numerical stability, and since the two dimensional schemes considered in this report are composed of one dimensional operators which have been analyzed in an earlier section, the two dimensional schemes may be considered to be stable, subject of course to the one dimensional stability requirements.

The two dimensional test problem is the following. A square mesh is chosen with $\Delta x=\Delta y=1$, and with 50 zones in each direction. In all cases Δt is 0.5. A velocity field corresponding to solid rotation is generated by $u=-(y-y_0)/25$ and $v=(x-x_0)/25$, where u and v are the Cartesian velocity components, the angular velocity of this flow field thus being 0.04 per unit time, and the flow field circling around the point $(x_0,\ y_0)=(25,\ 25)$. The dependent variable, Ψ, is set constant everywhere except in a region surrounding the point (25, 10) where it composes a right circular cone with base radius 5 and height 1.

Figure 14 is a plot of initial conditions for the two dimensional tests. The short line segments represent velocity vectors, giving the direction and (scaled) magnitude of the (constant in time) velocity field. There is one velocity vector plotted in each zone. The circular pattern in the lower center is a plot of level lines of $\psi(x, y)$. Since the evaluation of ψ is governed only by advection, and the specified velocity field corresponds to solid rotation, the

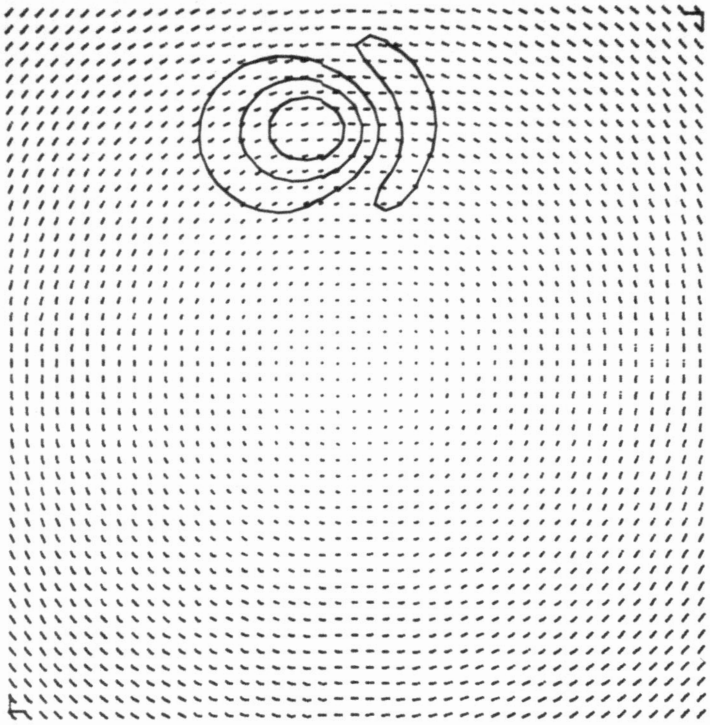

FIGURE 15.—Second order solution at cycle 175.

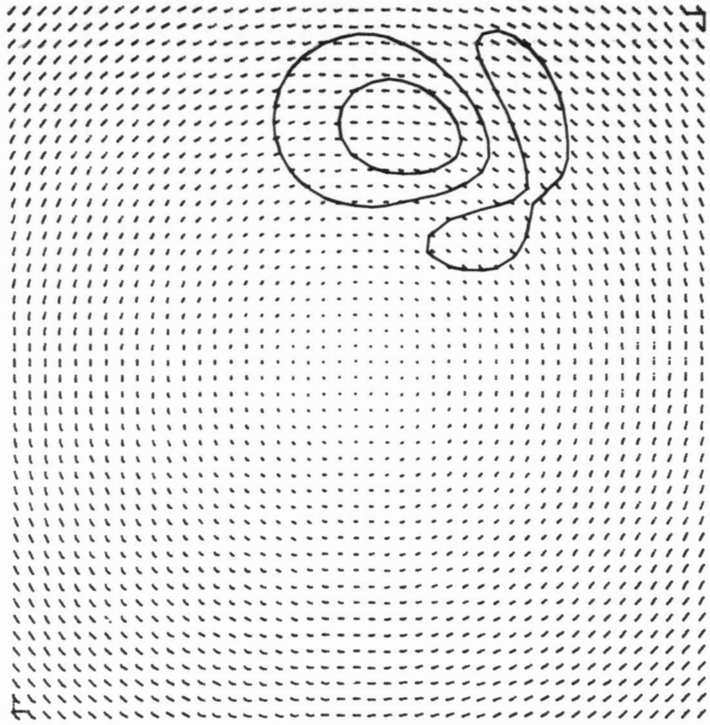

FIGURE 17.—Second order solution at cycle 475.

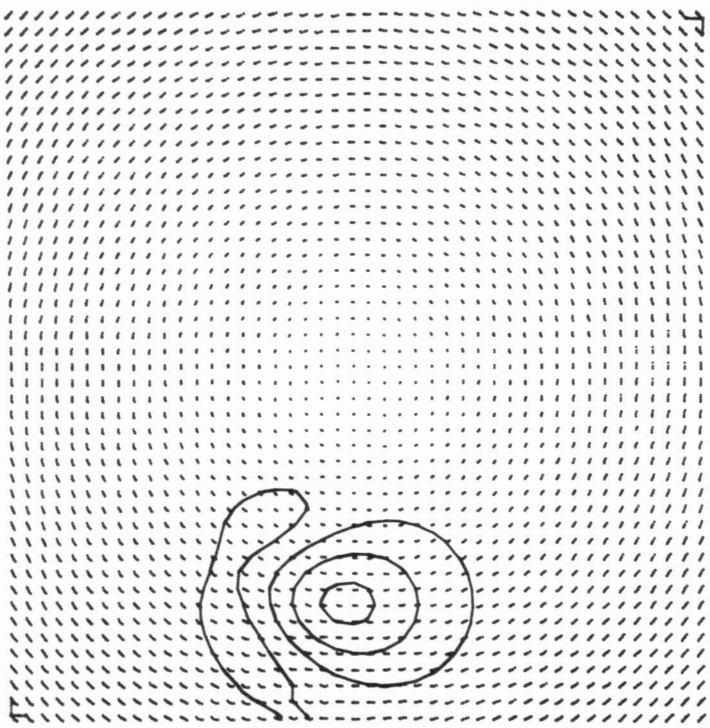

FIGURE 16.—Second order solution at cycle 325.

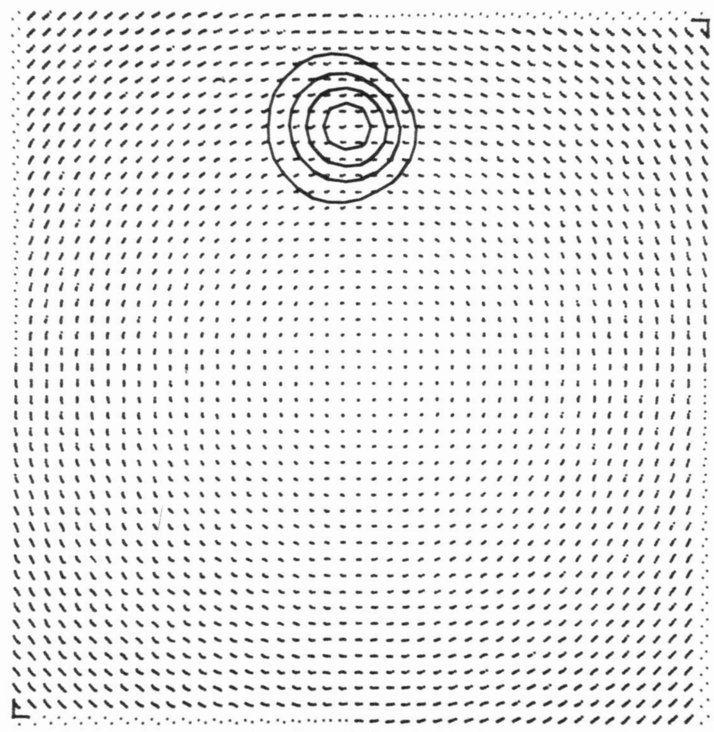

FIGURE 18.—Fourth order solution at cycle 475.

circular pattern should migrate in a circle of constant radius, in a counter-clockwise direction around the center of the mesh.

Figures 15, 16, and 17 show the resulting ψ field at cycles 175, 325, and 475, respectively, when the difference equation is the second order scheme described. These plots are essentially snapshots taken of contour lines of the ψ field after it has been rotated by the velocity field approximately through π, 2π, and 3π radians, respectively.

Figures 18 and 19 show the resulting ψ field after 475 and 950 cycles (rotations through 3π and 6π radians), respectively, when the difference equation is the fourth order advection scheme.

These plots are all unretouched CRT plots made with existing plotting routines (which use linear interpolation

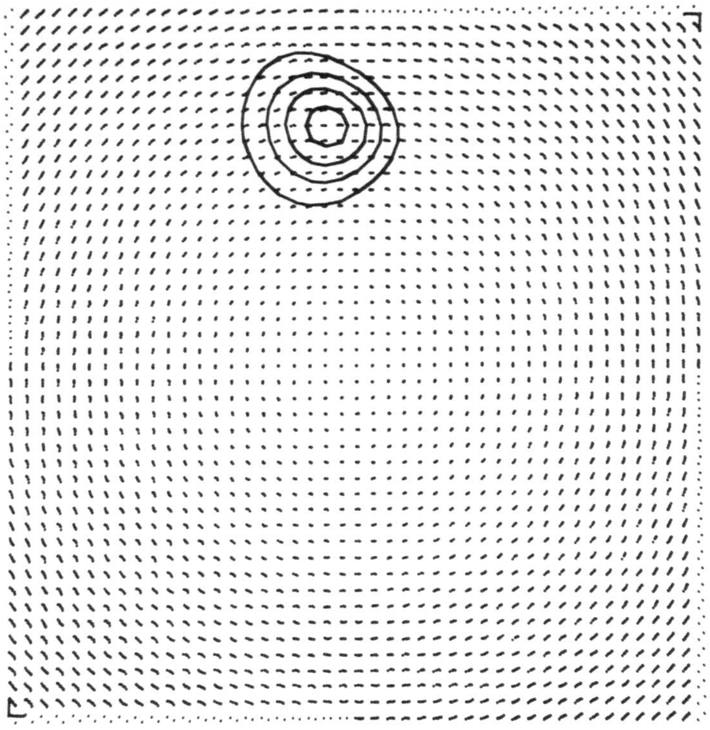

FIGURE 19.—Fourth order solution at cycle 950.

procedures) on the computer, during the execution of a problem.

Both of these calculations were repeated using the appropriate equations in conservation form, but the results were not qualitatively different from results obtained from the advection forms. Since the two dimensional test case involves nondivergent flow, this result is not too surprising.

Test calculations have also been made in which the difference equation was not split into separate sweeps, corresponding to

$$\psi^{N+1}=(I-A-B)\psi^{N}.$$

These tests all showed serious distortions in the ψ field after a rotation of approximately $\pi/2$ radians. This instability was pointed out, for the quadratic advection scheme, by Leith [5].

It is apparent from these crude and qualitative results, that for problems in which advection is important, fourth order schemes offer quite an increase in accuracy over second order schemes. It should be pointed out, however, that in practice velocity fields are usually not constant in time, but rather are coupled in a nonlinear manner to the quantity being advected. The final evaluation of the usefulness of these higher order advection schemes can be made only after they are tested in more realistic hydrodynamic models.

A short computer-generated motion picture has been produced that displays the second and fourth order two dimensional results.

8. SUMMARY

Finite difference approximations to advection have been examined for the color equation in conservation form and in advection form. The numerical properties of the one dimensional formulations have been displayed by plotting the errors in amplitude and phase due to the finite difference approximations. From these contour plots, it is seen that in general, amplitude and phase errors decrease (at the expense of increased computing costs) when higher order estimates of derivatives are used. Numerical experiments confirm this and also show that the conservation form is to be preferred over the advection form.

Approximations to time dependent advection in two space dimensions based on Marchuk's method of fractional time steps have also been examined. Test calculations show a dramatic improvement in results obtained from a fourth order scheme as compared with results from a second order scheme. These particular results are independent of whether the equation is in advection or conservation form, but it is felt that, as in one dimension, the more accurate solution will come from equations in conservation form.

REFERENCES

1. K. Bryan, "A Scheme for Numerical Integration of the Equations of Motion on an Irregular Grid Free of Nonlinear Instability," *Monthly Weather Review*, vol. 94, No. 1, Jan. 1966, pp. 39–40.
2. S. Burstein, "Finite Difference Calculations for Hydrodynamic Flows Containing Discontinuities," *Journal of Computational Physics*, vol. 1, No. 2, 1967, pp. 198–222.
3. W. P. Crowley, "Second Order Numerical Advection," Journal of Computational Physics, vol. 1, No. 4, June 1967, pp. 471–484.
4. P. Lax and B. Wendroff, "Difference Schemes for Hyperbolic Equations With High Order of Accuracy," *Communications on Pure and Applied Mathematics*, 1964, pp. 381–398.
5. C. E. Leith, "Numerical Simulation of the Earth's Atmosphere," *Methods in Computational Physics*, vol. 4, Academic Press, New York, 1965, pp. 1–28.
6. G. I. Marchuk, "Teoreticheskaia Model' Prognoza Pogody," [Theoretical Model for Weather Forecasting], *Akademiia Nauk SSSR, Doklady*, vol. 155, No. 5, Apr. 1964, pp. 1062–1065.
7. A. J. McConnell, *Applications of Tensor Analysis*, Dover Publications Reprint, New York, 1947, 318 pp.
8. W. Noh, "Cel: A Time-Dependent Two-Space Dimensional Coupled Eulerian-Lagrange Code," *Methods in Computational Physics*, vol. 3, Academic Press, New York, 1964, pp. 117–179.
9. R. D. Richtmyer, *Difference Methods for Initial Value Problems*, Interscience Publishers, New York, 1957, 238 pp.
10. R. D. Richtmyer, "A Survey of Difference Methods for Non-Steady Fluid Dynamics," *NCAR Technical Note 63-2*, National Center for Atmospheric Research, Boulder, Colorado, 1962, 325 pp.
11. K. Roberts and N. Weiss, "Convective Difference Schemes," *Mathematics of Computations*, vol. 20, No. 94, 1966, pp. 272–299.

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